Announcements

1) Error in HW, #5 (last question), absolute values changed to d_1, d_2 where appropriate

2) Review next week

Proposition: (functions that differ by a constant)

If $f, g : [a,b] \to \mathbb{R}$ and are differentiable on $[a,b]$, then if

$$\underline{f'(x) = g'(x)} \quad \forall x \in [a,b],$$

$\exists\, c \in \mathbb{R}$ with

$$\underline{f(x) = g(x) + c} \quad \forall x \in [a,b].$$

proof: Let $h(x) = f(x) - g(x)$.

Then h is differentiable on $[a,b]$, so for any $y \in [a,b]$, $\exists\ c \in (a,b)$ with

$$h'(c) = \frac{h(y) - h(a)}{y - a} \quad (y \neq a)$$

by the Mean Value Theorem,

But $h'(c) = f'(c) - g'(c)$
$= 0$

since we have $f' = g'$ on $[a,b]$.

Then

$$0 = \frac{h(y)-h(a)}{y-a}, \text{ and } y \neq a$$

implies $h(y)-h(a)=0$,

so $h(y)=h(a)$.

Substituting,

$$f(y)-g(y) = \underbrace{f(a)-g(a)}_{c} \quad \forall y \in [a,b]$$

and hence,

$$f(y) = g(y) + c. \quad \square$$

Corollary: If f is differentiable on $[a,b]$ and $f'(x) = 0$ for all $x \in [a,b]$, then $\exists\, c \in \mathbb{R}$ with

$$f(x) = c.$$

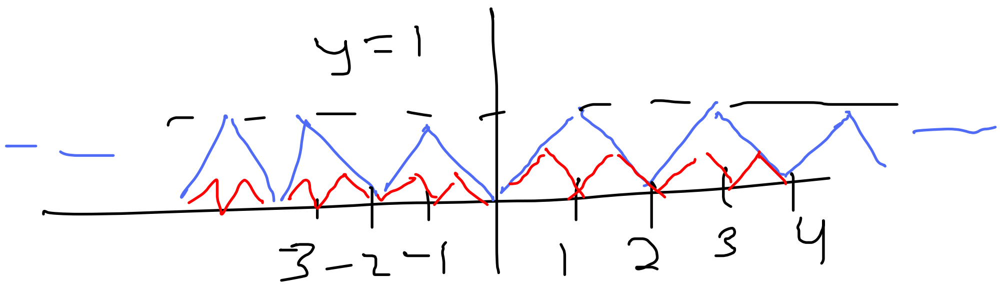
Proof: Let $g(x) = 0$ in the previous proposition $\square$

A Continuous, Nowhere Differentiable Function

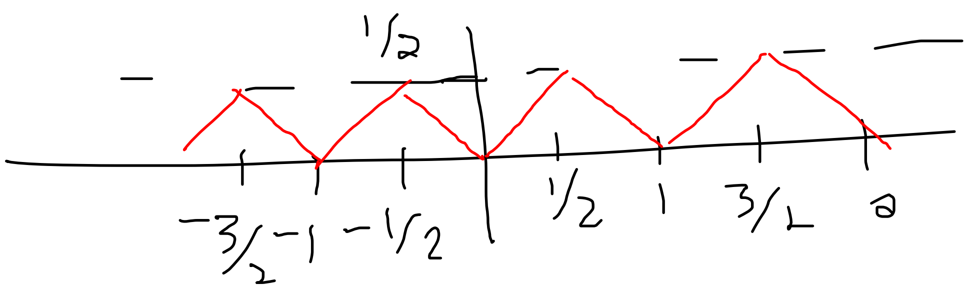
Idea: Start with $f(x) = |x|$, replicate at successively smaller intervals.

Picture

Start off with $h(x)$



Create $h_1(x)$ by shrinking h by a factor of 2



Make h_2 by shrinking h_1 by a factor of 2. Continue ~

Define

$$h(x) = \begin{cases} |x|, & -1 \le x \le 1 \\ |x| - 2n, & 2n-1 \le x \le 2n \\ & n \in \mathbb{Z} \end{cases}$$

$$h_1(x) = \frac{1}{2} h(2x)$$

$$h_2(x) = \frac{1}{4} h(4x) = \frac{1}{2} h_1(2x)$$

In general,

$$h_n(x) = \frac{1}{2^n} h(2^n x)$$

Facts about h

1) h is continuous on $\mathbb{R}$

2) $h(2n) = 0 \quad \forall\, n \in \mathbb{Z}$

3) $0 \leq h(x) \leq 1 \quad \forall\, x \in \mathbb{R}$

4) h fails to be differentiable for all $n \in \mathbb{Z}$

Define $h_0(x) = h(x)$ and

$$g_m(x) = \sum_{n=0}^{m} h_n(x)$$

$$= \sum_{n=0}^{m} \frac{1}{2^n} h(2^n x)$$

Set $\boxed{g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)}$

$\forall\ x \in \mathbb{R}$. Claim: g is

continuous but nowhere

differentiable!

Note. Since h_n is continuous $\forall\, n \in \mathbb{N} \cup \{0\}$,

g_m is continuous $\forall\, m \in \mathbb{N} \cup \{0\}$

The series defining g makes sense because

$$\sum_{n=0}^{\infty} \left| \frac{1}{2^n} h(2^n x) \right| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} \quad (0 \leq h \leq 1)$$

$$= 2$$

So the series is absolutely convergent $\forall\, x \in \mathbb{R}$, hence convergent

Moreover:

$$\left| g_m(x) - g(x) \right|$$

$$= \left| \sum_{n=m+1}^{\infty} \frac{1}{2^n} h(2^n x) \right|$$

$$\leq \sum_{n=m+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m}$$

regardless of the value of x!

If $\varepsilon > 0$, choose m so that $\frac{1}{\varepsilon} < 2^m$. Then

$$\left| g_m(x) - g(x) \right| < \varepsilon.$$

regardless of x!

Using this fact, we show g is continuous.

let $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Then

$$|g(x) - g(y)| =$$

$$= |g(x) - g_m(x) + g_m(x) - g(y)|$$

$$\leq |g(x) - g_m(x)| + |g_m(x) - g(y)|$$

$$= |g_m(x) - g_m(y) + g_m(y) - g(y)|$$

$$\leq |g_m(x) - g_m(y)| + |g_m(y) - g(y)|$$

Then

$|g(x)-g(y)|$

$\leq |g(x)-g_m(x)| + |g_m(x)-g_m(y)| + |g_m(y)-g(y)|$

Let $\varepsilon>0$. Choose $m\in\mathbb{N}$ so that

$|g(x)-g_m(x)| < \frac{\varepsilon}{3} \quad \forall\ x\in\mathbb{R}$

Since g_m is continuous,

$\exists\ \delta>0$ such that

$|g_m(x)-g_m(y)| < \frac{\varepsilon}{3}$ when

$|x-y|<\delta$. With this m, δ,

we get $|g(x)-g(y)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$

Show g is not differentiable at any $x \in \mathbb{R}$

Step 1: $x = \dfrac{p}{2^m}$ for some

$p \in \mathbb{Z}$ and $m \in \mathbb{N} \cup \{0\}$

(x is dyadic).

At value m, g_m is not differentiable at precisely these points!

But $g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$,

So

$$g\left(\frac{p}{2^m}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} h\left(\frac{2^n p}{2^m}\right)$$

Remember! h is zero at all even integers

If $n \geq m+1$, then

$\frac{2^n p}{2^m} = 2^{n-m} p$ is an

even integer since $p \in \mathbb{Z}$ $n-m \geq 1$!

Therefore, $h\left(\frac{2^n p}{2^m}\right) = 0$ for all such n.

Then $g\left(\frac{p}{2^m}\right) = \sum_{n=0}^{m} \frac{1}{2^n} h\left(\frac{2^n p}{2^m}\right)$

$= g_m\left(\frac{p}{2^m}\right)$

which is not differentiable at $\frac{p}{2^m}$! There, g is not differentiable at $\frac{p}{2^m}$ . . .

Step 2: arbitrary x not dyadic

Believe (no, check for yourself) that

$\forall x \in \mathbb{R}$, x not dyadic,

$\exists p_m \in \mathbb{Z}$ such that

$$\frac{p_m}{2^m} < x < \frac{p_m+1}{2^m} \quad \forall m \in \mathbb{N} \cup \{0\}$$

(dyadic expansion of x)

- If $x_m = \frac{p_m}{2^m}$, $y_m = \frac{p_m + 1}{2^m}$,

$x_m \to x$, $y_m \to x$ as $m \to \infty$.

- $| g'_{m+1}(x) - g'_m(x) | = 1$

- $$\frac{g(y_m) - g(x)}{y_m - x} < g'_m(x) < \frac{g(x_m) - g(x)}{x_m - x}.$$

Assume, by contradiction, that $g'(x)$ exists, then taking $\lim_{m\to\infty}$ in previous inequality, by Squeeze Theorem,

$$\lim_{m\to\infty} g_m'(x) = g'(x).$$

But $|g_{m+1}'(x) - g_m'(x)| = 1$, contradiction! That's all!